

Classical kinetical Bose gas

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An autocatalytic reaction combined with spontaneous creation and annihilation processes of particles are studied in a quantum formalism of the master equation in a lattice gas representation with unrestricted occupancy. In case the system is activated by a linear coupling to a heat bath the problem can be solved exactly and the stationary particle density follows the Bose distribution. The relation to spin-flip processes with a restricted occupancy is discussed. Different from those processes the relaxation time and the density fluctuation increase in the high-temperature limit. On a small scale the mutual interaction between the particles is relevant. While in case of a repulsive interaction the stationary solution becomes unstable against short wavelength fluctuations, an attractive interaction leads to an instability for long wavelength fluctuations. The system decays in domains, the size of which can be estimated as a function of temperature and interaction strength. The model is also appropriate to study the growth of open bacterial colonies under the influence of a competitive interaction between the species.

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I. INTRODUCTION

Many systems behave on the phenomenological level essentially randomly and, therefore, other approaches for the theoretical treatment have to be employed. The randomness, resulting from stochastic forces or be intrinsic in the underlying microscopic theory, inevitably leads to the description of such systems in terms of probabilities and expectation values [1,2]. The time development of the probability is usually found using a master equation. The past years have seen an exciting new development based on the observation [3] of the close relationship between the Markov generator of the master equation and a time evolution operator acting on a many-particle Fock space [4,5]; for some recent reviews compare [6,7]. The new insight has led to a series of remarkable exact solutions for the stochastic dynamics of interacting particle systems; for a recent overview see Ref. [8]. Despite exact results the mentioned method has also been fruitful in an approximating description of other models such as the facilitated kinetic Ising system as a candidate for glassy systems [9–11] or in branching and annihilation random walks [12]. Whereas the original paper [3], see also Ref. [13–18], is concerned with a mapping of the master equation to a representation in terms of second-quantized bosonic operators great progress for exact solvable models had been achieved by mapping to spin-1/2 Pauli operators [8]. This mapping to spin systems applies to processes where each lattice site can be occupied by only a finite number of particles. Physically, this restriction may be hard-core constraints or fast on-site annihilation processes. Obviously, such a mapping simulates the exclusion principle for classical lattice models within a cellular automata. The dynamics of systems under exclusion is often described by spin-flip and spin-exchange processes. However, the corresponding counterparts in terms of Bose operators are failed. In the present paper we show that the autocatalytic reaction $A \rightarrow 2A$ combined with a spontaneous creation and annihilation

of particles $\emptyset \rightleftharpoons A$ is able to play the role of that counterpart in case of an unrestricted occupancy at each lattice point. Thus, the situation is related to the quantum Bose statistics. However, let us point out that our approach is concerned completely with stochastically classical systems obeying a master equation. The Fock-space description is a mathematical tool applied for a system far from equilibrium that is coupled to a heat bath and where the particles are subject to a pair interaction.

The model could also be of interest for the growth of open bacterial colonies. The colony grows up when a bacterium is multiplied inside the system or when it is incorporated from outside. The unlimited increase of the colony is prevented by external parameters such as the nutrient concentration and the temperature. An additional mutual interaction between the bacteria is also able to favor growth or death processes inside the colony. The model can further be interpreted as a model for “precipitation” of matter in space. Starting with the empty space coupled to a particle reservoir at temperature T the density will be locally enhanced by a random deposition of particles and a simultaneous removal of particles with different rates. Because the occupation number per lattice site is unrestricted the averaged density is likewise arbitrary. When the deposition and the removal rates are temperature dependent via an Arrhenius law, then the stationary distribution is simply the Bose occupation number despite the particles offering a classical behavior. Apparently, the precipitation of matter in space is supported by an attractive interaction whereas a repulsive interaction is a competing process compared to the kinetic processes discussed above.

In the present paper such kinds of models are discussed by means of a Fock-space representation of the underlying master equation. Moreover the differences from the corresponding approach in terms of spin operators is studied in detail. The stability of the stationary solution with respect to the spatial fluctuation, originated by an interaction of the particles, is analyzed. The paper is organized as follows.

First the model is introduced based on the conventional master equation approach. The equivalence of the evolution equation to a quantum formulation in terms of Bose operators will be studied in Sec. III where also the differences from spin flip are studied. The influence of an additional interaction of the particles is analyzed in Sec. IV whereas the stability of the system against fluctuation is considered in Sec. V. In the last section the relevance of the approach for reactions, open bacterial colonies, and the precipitation of matter are discussed.

II. MODEL AND MASTER EQUATION APPROACH

A. The autocatalytic reaction

Let us consider a chemical autocatalytical reaction that is attended by a spontaneous creation and annihilation process,



Whereas the creation of a particle A and its subsequent doubling to $2A$ should be realized with the same rate γ , the spontaneous decay $A \rightarrow \emptyset$ is fulfilled with the rate λ . The master equation for these processes is given by

$$\begin{aligned} \partial_t p(n,t) &= \lambda(n+1)p(n+1,t) + \gamma n p(n-1,t) \\ &\quad - [\lambda n + \gamma(n+1)]p(n,t). \end{aligned} \quad (2)$$

Here $p(n,t)$ is the probability density that n particles are present at time t . The prefactors n or $n+1$, respectively are originated by the number of combinations for the reactions. Eq. (2) can be solved by a probability generating function [1]

$$G(z,t) = \sum_m z^m p(m,t),$$

which obeys

$$\partial_t G = \partial_z G(1-z)[\lambda - \gamma z] - \gamma(1-z)G. \quad (3)$$

The time independent solution is simply $G_s(z) = (\lambda - \gamma)(\lambda - \gamma z)^{-1}$ from which one can obtain the stationary probability density

$$p_s(n) = \frac{\lambda - \gamma}{\lambda} \left(\frac{\gamma}{\lambda} \right)^n. \quad (4)$$

Hence the averaged occupation number in the stationary limit reads

$$\langle n \rangle = \frac{\gamma}{\lambda - \gamma} = \frac{1}{\exp(\varepsilon/T) - 1}, \quad (5)$$

where the last relation holds when the rates λ and γ are introduced via an Arrhenius law, i.e.,

$$\lambda = \mu \exp(\varepsilon/2T), \quad \gamma = \mu \exp(-\varepsilon/2T) \quad (6)$$

with a positive activation energy $\varepsilon > 0$. The last relation is discussed in more detail in Sec. II B. The equilibrium en-

tropy $S = \langle \ln p_s(n) \rangle$ is nothing else as $S = (1 + \langle n \rangle) \ln(1 + \langle n \rangle) - \langle n \rangle \ln \langle n \rangle$ which is well known from quantum statistics. The time dependent probability density can be derived using the solution of Eq. (3). We get

$$G(z,t) = \frac{\lambda - \gamma}{\lambda - \gamma z} \Psi \left(\frac{z-1}{\lambda - \gamma z} e^{-t/\tau} \right), \quad (7)$$

with the relaxation time $\tau = (\lambda - \gamma)^{-1}$. $\Psi(Y)$ is an arbitrary function that is determined through the initial density $\langle n(0) \rangle$ and all its moments. In the case that the initial density is zero and the higher order moments are uncorrelated, it results in

$$\Psi(Y) = \frac{1}{1 + \gamma Y}.$$

From here and Eq. (7) we obtain the probability density to be

$$p(n,t) = p_s(n) \frac{(1 - e^{-t/\tau})^n}{[1 - (\gamma/\lambda)e^{-t/\tau}]^{n+1}}. \quad (8)$$

Notice that we had not been successful in finding a closed expression of the function $\Psi(y)$ for different initial conditions. For comparison with the quantum formalism introduced in Sec. II B the first terms of a Taylor expansion of $\Psi(Y)$ are given:

$$\Psi(0) = 1, \quad \frac{\partial \Psi(0)}{\partial Y} = (\lambda - \gamma) \langle n(0) \rangle - \gamma,$$

$$\frac{\partial^2 \Psi(0)}{\partial^2 Y} = 2\gamma^2 + [(\lambda + \gamma)^2 - 4\gamma^2] \langle n(0) \rangle + (\lambda - \gamma)^2 \langle n^2(0) \rangle. \quad (9)$$

B. Quantum approach to nonequilibrium

In this section we introduce shortly the quantum formalism for the master equation that is formally written as

$$\partial_t p(\mathbf{n},t) = L' p(\mathbf{n},t), \quad (10)$$

where $p(\mathbf{n},t)$ is the probability that a certain configuration on a lattice characterized by a state vector $\mathbf{n} = (n_1, n_2, \dots, n_N)$ is realized at time t . Here, we consider the case of an unrestricted occupancy, i.e., $n_i = 0, 1, 2, \dots$. The occupation numbers n_i are considered as the eigenvalues of the particle number operator defined by creation operators a_i^\dagger or by annihilation operators a_i . The problem is to formulate the dynamics in such a way that the possible realizations for the occupation numbers are taken into account explicitly. The situation in mind can be analyzed in a seemingly compact form using the master equation in a quantum Hamilton formalism [2–5, 19] (for recent reviews, see Refs. [7, 8]). The dynamics is determined by the form of the evolution operator L' , specified below. Within that approach [3] the probability distribution $p(\mathbf{n},t)$ is related to a state vector $|F(t)\rangle$ in a Fock space according to $p(\mathbf{n},t) = \langle \mathbf{n} | F(t) \rangle$. The basic vectors $|\mathbf{n}\rangle$ are composed of the operators a_i^\dagger and a_i . Using the relation

$$|F(t)\rangle = \sum_{\mathbf{n}_i} p(\mathbf{n}, t) |\mathbf{n}\rangle, \quad (11)$$

the master Eq. (10) can be transformed into an equivalent one in a Fock space

$$\partial_t |F(t)\rangle = L |F(t)\rangle, \quad (12)$$

where the operator L' in Eq. (10) is mapped onto the operator $L = \sum |\mathbf{m}\rangle L'_{mn} \langle \mathbf{n}|$ in Eq. (12). It should be emphasized that the procedure is up to now independent on the realization of the basic vectors. Originally, the method had been applied for the Bose case [3,4,13,20]. Recently, an extension to restricted occupation numbers (two discrete orientations) was proposed [2,5,19]. Further extensions to p -fold occupation numbers [11] as well as to models with kinetic constraints [21] and to systems with two heat baths [22] are possible. As shown by Doi [3] the average of an arbitrary physical quantity $B(\mathbf{n})$ can be calculated by the average of the corresponding operator $B(t)$,

$$\langle B(t) \rangle = \sum_{\mathbf{n}_i} p(\mathbf{n}, t) B(\mathbf{n}) = \langle s | B | F(t) \rangle, \quad (13)$$

with the state function $\langle s | = \sum \langle \mathbf{n} |$. The evolution equation for an operator $B(t)$ now reads

$$\partial_t \langle B \rangle = \langle s | [B(t), L] | F(t) \rangle. \quad (14)$$

As the result of the procedure, all the dynamical equations governed by the classical problem are determined by the structure of the evolution operator L and the commutation rules of the operators.

III. ACTIVATED PROCESS

In this section we will apply the procedure introduced in the preceding section to find out the realization of the quantum evolution operator L_b for the master equation introduced by Eq. (2). To be more specific a d -dimensional lattice is assumed where each lattice site can be occupied by an arbitrary number of particles. Furthermore, the system is coupled to a particle reservoir with an infinite number of particles. The dynamics consists of random processes where in a stochastic manner particles will be added to the system or removed from it according to the chemical reactions in Eq. (1) or alternatively as birth and death processes in bacterial colonies. The second-quantized evolution operator based on Eqs. (2) and (11) is given by

$$L_b = \sum_j [\lambda(a_j - n_j) + \gamma(a_j^\dagger - 1)(1 + n_j)]. \quad (15)$$

Here, the particle number operator $n_j = a_j^\dagger a_j$ is introduced the eigenvalues of which are unrestricted. Hence, the operators a and a^\dagger satisfy the commutation rules $[a_i, a_j^\dagger] = \delta_{ij}$. L_b is the counterpart of the corresponding operator L_f that describes spin-flip processes. In case of a two state model the evolution operator is written in the form, compare, for instance [8],

$$L_f = \sum_j [\lambda(d_j - D_j) + \gamma(d_j^\dagger - (1 - D_j))], \quad (16)$$

where the operators d and d^\dagger fulfill the commutation rules of Pauli operators $[d_i, d_j^\dagger] = \delta_{ij}(1 - 2D_i)$, $D_i = d_i^\dagger d_i$. Applying Eq. (14) the evolution equation for the averaged densities satisfies

$$\begin{aligned} \partial_t \langle n_r \rangle &= -\lambda \langle n_r \rangle + \gamma \langle 1 + n_r \rangle, \\ \partial_t \langle D_r \rangle &= -\lambda \langle D_r \rangle + \gamma \langle 1 - D_r \rangle. \end{aligned} \quad (17)$$

The evolution equation reflects both competing processes. Whereas the first term offers the loss due to the annihilation process the second one gives rise to the gain due to the reaction. The stationary solutions are given by

$$\langle n_r \rangle_s = \frac{\gamma}{\lambda - \gamma}, \quad \langle D_r \rangle_s = \frac{\gamma}{\lambda + \gamma}. \quad (18)$$

Because the kinetic process does not support any spatial ordering the stationary solution is a homogeneous one. In case of a Bose system the kinetic coefficients should obey $\lambda > \gamma$ which guarantees a stable stationary solution due to the competition among birth and death processes. If the kinetic coefficients follow an Arrhenius law introduced by Eq. (6) the system evolves to a physical accessible positive density. Inserting Eq. (6) into Eq. (18) the stationary solutions obey indeed the Bose or the Fermi distribution, respectively.

$$n_s \equiv \langle n_r \rangle_s = \frac{1}{\exp(\varepsilon/T) - 1}, \quad \langle D_r \rangle_s = \frac{1}{\exp(\varepsilon/T) + 1}. \quad (19)$$

Notice that in the case when the term $1 + n_j$ is absent in Eq. (15) the Boltzmann distribution is realized.

As it had been demonstrated in Ref. [9] for spin systems, the Arrhenius law, Eq. (6), can be incorporated in the analytical formulation by assuming a linear coupling to a heat bath at temperature T . Following the line proposed in Ref. [9] we generalize the procedure to an unrestricted occupancy. The evolution operator L_b , Eq. (15), is modified in terms of Bose operators at finite temperatures in the following form:

$$\begin{aligned} L_b &= \mu \sum_j [(1 - a_j^\dagger) \exp(-\beta H/2) a_j \exp(\beta H/2) \\ &\quad + (a_j^\dagger a_j - a_j) \exp(-\beta H/2) a_j^\dagger \exp(\beta H/2)], \end{aligned} \quad (20)$$

where the Hamiltonian H includes a linear coupling to a heat bath and further a mutual interaction between the particles. It should also be expressed in terms of Bose operators and moreover in a diagonalized form (for specification see below). As usual $\beta^{-1} = T$ is the temperature of the external heat bath, the parameter μ determines a microscopic time scale that cannot be determined within a mesoscopic approach. During the time μ^{-1} one particle can be created and annihilated. Therefore it seems to be reasonable that μ is independent of temperature. Notice that Eq. (20) is in accor-

dance with the principle of detailed balance. In the simplest case of a linear coupling to a heat bath, the Hamiltonian is given by

$$H_0 = \sum_j \varepsilon n_j,$$

where ε is a single particle energy or an external field that is assumed to be positive. Using the commutation rules for Bose operators it can be checked easily that Eq. (17) has to be replaced by

$$\mu^{-1} \partial_t \langle n_r \rangle = -\exp(\beta\varepsilon) \langle n_r \rangle + \exp(-\beta\varepsilon) \langle 1 + n_r \rangle. \quad (21)$$

This relation is in accordance with the replacement suggested in Eq. (6). The parameter $\lambda = \mu \exp(\varepsilon/2T)$ characterizes the annihilation rate. The death process dominates the behavior of the system for low temperatures. For instance, the bacterial colony dies out. The rate $\gamma = \mu \exp(-\varepsilon/2T)$ describes the growth process that is suppressed at low temperatures. At high temperatures both rates tend to be of the same order. For $T \rightarrow \infty$ the doubling reaction $A \rightarrow 2A$ is the dominating one. As the result of the competing processes the system evolves to the stationary solution Eq. (18) that is indeed the Bose distribution as argued above. The solution of Eq. (21) is trivial and leads to an exponential decay with a relaxation time

$$\mu \tau_b = \frac{1}{2 \sinh\left(\frac{\varepsilon}{2T}\right)}, \quad (22)$$

which is in accordance with the previous result in Eq. (7). Proceeding in the same manner for the spin system coupled to a heat bath (which leads to the Fermi distribution) the exponential decay gives rise to a relaxation time of the form

$$\mu \tau_f = \frac{1}{2 \cosh\left(\frac{\varepsilon}{2T}\right)}. \quad (23)$$

Both relaxation times differ in the high-temperature limit $\varepsilon/T \ll 1$ whereas they merge together in the low-temperature limit $\varepsilon/T \gg 1$. In particular, one gets for high temperatures

$$\begin{aligned} \mu \tau_b &\approx \frac{T}{\varepsilon} - \frac{\varepsilon}{24T}, \\ \mu \tau_f &\approx \frac{1}{2} - \left(\frac{\varepsilon}{4T}\right). \end{aligned} \quad (24)$$

The different behavior of the relaxation times τ_b and τ_f is obviously due to different behavior of the averaged occupation numbers. While in case of an unrestricted occupancy each lattice site is filled up with an arbitrary number of particles for high temperatures the averaged occupation number for a spin system remains restricted. Consequently, the relaxation time in units of the elementary time scale μ goes to 1/2

for infinite temperature for the spin system. In the Bose case the particle density increases and, therefore, the relaxation time increases too. In the low-temperature limit we get

$$\tau_b = \tau_f \approx \exp[-\varepsilon/(2T)].$$

The situation in mind is apparently inverse to the corresponding one in conventional quantum statistics. In that case Fermi and Bose distribution merge together for high temperatures resulting in a Boltzmann distribution and differ in the low-temperature case. Here, we consider particles that are subject to stochastic flip processes under the exclusion principle or alternatively, the creation or annihilation processes with an unrestricted occupation number per lattice site. The stochastic process is further restricted for low temperatures. Hence one should not expect a significant difference for classical systems with and without exclusion. However, for high temperature each lattice site can be occupied with an optimal number of particles. In the Fermi case that is realized with the value $\lim_{T \rightarrow \infty} \langle D \rangle_s = 1/2$ whereas in the Bose case an infinite occupation is allowed.

The approach allows us also to determine easily the local fluctuations. Defining

$$w_i = \langle n_i^2 \rangle - \langle n_i \rangle^2, \quad (25)$$

we find the evolution equation of the fluctuations that leads to a stationary solution

$$w_s = \frac{1}{4 \sinh^2(\varepsilon/(2T))}. \quad (26)$$

For high temperatures the fluctuation strength increases with T , $w_s \propto T^2$ whereas for low temperatures the local density fluctuations decay exponentially $w_s \approx \exp(-\varepsilon/T)$. Notice that this relation can also be derived using Eqs. (7) and (9).

IV. INTERACTION

As stressed recently [23,24] for the completely different problem of a catalytic reaction, the lateral interactions between adsorbed particles should affect the behavior of the system. Therefore we demonstrate in this section for our model how the quantum approach for a classical stochastic system can be extended by the inclusion of interaction between the reactants. Whereas the original evolution operator, Eq. (15), is a local one, a short range density-density interaction may lead to a further competitive situation between the bacteria, for instance, due to a limited nutrient concentration. The reactants can be affected likewise by an interaction. As the result of such an interaction one should expect the occurrence of spatial correlations. If the particles are subject to an attractive interaction the autocatalytic reaction, or the growth of the colony or the precipitation process could be supported whereas in the opposite case of a repulsive force the decay or death processes are favored. The simplest realization of an interaction can be defined via a density-density correlation with the energy

$$H = \sum_j \varepsilon_j n_j + \sum_{i,j} J_{ij} n_i n_j, \quad (27)$$

where ε_j is the single particle energy, already discussed before, and J_{ij} with $i \neq j$ is the coupling strength between the z nearest neighbors. First let us estimate the influence of the interaction for high temperatures. If n is the averaged particle density we find from Eq. (19) $n \approx T/\varepsilon$. From here we conclude

$$\frac{H}{T} \approx 1 \pm \frac{T}{T_0} \quad \text{with} \quad T_0 = \frac{\varepsilon^2}{2|J|z}, \quad (28)$$

where the plus sign belongs to the repulsive and the minus sign to the attractive interaction. The characteristic temperature T_0 will be discussed in the following section.

In case of the inclusion of interaction the resulting evolution equation is a nonlinear one,

$$\begin{aligned} \mu^{-1} \partial_t \langle n_r \rangle &= -\langle n_r U_r \rangle \exp(\kappa_r) + \langle (1 + n_r) U_r^{-1} \rangle \exp(-\kappa_r) \\ \text{with } \kappa_r &= \frac{\varepsilon_r}{2T}, \quad U_r = \exp\left(\frac{1}{T} \sum_{j(r)} J_{rj} n_j\right), \end{aligned} \quad (29)$$

where $j(r)$ means the sum over all neighbors to the lattice point r . Because we are interested in long range correlations in space and time let us neglect all correlations on the microscopic scale. We expect that the system develops a strong correlated behavior in the large time limit and on a higher resolved spatial scale $|\mathbf{x}| \gg l$, where l is a lattice size that will be set to be $l = 1$. Therefore we can write the evolution equation for the averaged density denoted by $\langle n_r(t) \rangle = n(\mathbf{x}, t)$ in the form

$$\begin{aligned} \mu^{-1} \partial_t n(\mathbf{x}, t) &= -2n(\mathbf{x}, t) n h \left[\kappa(\mathbf{x}) + \frac{J}{T} (\nabla^2 n(\mathbf{x}, t) + z n(\mathbf{x}, t)) \right] \\ &+ \exp\left\{ -\kappa(\mathbf{x}) - \frac{J}{T} [\nabla^2 n(\mathbf{x}, t) + z n(\mathbf{x}, t)] \right\}. \end{aligned} \quad (30)$$

Here we have used the continuum approximation taking into account only the lowest order of a gradient expansion, z is the number of nearest neighbors and J is the interaction strength the sign of which determines the behavior of the system in a crucial manner. For instance, in the weak interaction limit $J/Tn_s \ll 1$ one finds that Eq. (30) leads always to a negative diffusion coefficient for a repulsive interaction. In the opposite case of an attractive interaction the effective diffusion coefficient is always positive and, therefore, that kind of interaction supports the precipitation process. The general case is considered in the following section.

V. STABILITY

Here we study the stability of the stationary solution against spatial fluctuations for an arbitrary interaction strength but with a fixed activation energy $\varepsilon = \text{const}$. As stressed before, the solution of Eq. (30) depends strongly on

the sign of the interaction strength J . Whereas for $J > 0$ the reactive process should be supported at least for short distances, a repulsive interaction $J < 0$ will lead to an unstable behavior. The situation in mind can be discussed based on Eq. (30) by using linear stability analysis. Performing Fourier transformation and making the ansatz,

$$n(\mathbf{k}, t) = n_s + y(\mathbf{k}, t),$$

we find from Eq. (30) an exponential behavior $y \propto \exp[-\mu\Omega(\mathbf{k})t]$ where the fluctuation mode obeys the following dispersion relation:

$$\Omega(\mathbf{k}) = r + Dk^2, \quad (31)$$

where the mass r and the diffusion parameter D are given by

$$\begin{aligned} r &= 2 \sinh\left(\frac{E^{(m)}}{2}\right) + \frac{Jz}{T} \sinh^{-1}\left(\frac{E^{(m)}}{2}\right), \\ D &= -\frac{J}{T} \sinh^{-1}\left(\frac{E^{(m)}}{2}\right) \quad \text{and} \quad E^{(m)} = \frac{\varepsilon + 2Jzn_s}{T}. \end{aligned}$$

The stationary value n_s fulfills a self-consistent mean-field-like equation, see also Eq. (27),

$$n_s = \frac{1}{\exp E^{(m)} - 1}, \quad (32)$$

which is physically accessible if $E^{(m)} > 0$. Notice that the results are not restricted to a weak interaction. Therefore in case of an arbitrary repulsive interaction $J > 0$ the system becomes unstable against spatial fluctuations at $k = k^* = \sqrt{r/D}$. Accordingly the system decays in small regions the size of which is given by

$$l^* = 2\pi \sqrt{\frac{J}{2T \sinh^2(E^{(m)}/2) + Jz}}. \quad (33)$$

The short range repulsive interaction leads to an instability on a length scale $l \leq l^*$, whereas on a scale $l > l^*$ the system remains stable. The situation is complete different in case of an attractive interaction $J < 0$. Since the diffusion parameter D is always positive that guarantees the stability against short wavelength fluctuations, the behavior of the system is dictated by the sign of the mass term r in Eq. (31). From Eq. (32) one concludes that the stationary solution n_s is accessible at finite temperatures only in the low density limit given by

$$0 < n_s < \frac{T_0}{\varepsilon} \quad \text{with} \quad T_0 = \frac{\varepsilon^2}{2|J|z}. \quad (34)$$

A violation of that condition for a high density leads immediately to a density collapse due to the attractive interaction. In particular, this case would be realized for a zero activation energy. Now, we study the dispersion relation for the fluctuating mode, Eq. (31), within the limited area of validity defined by Eq. (34). To that aim let us rewrite Eq. (31) as

$$n_s = \frac{2|J|z \exp(E^{(m)}/2)}{Tn_s} (n_{s_+} - n_s)(n - n_{s_-})$$

$$\text{with } 2n_{s_{\pm}} = \pm \sqrt{1 + \frac{4TT_0}{\varepsilon^2}} - 1. \quad (35)$$

Equations (34) and (35) offer a positive mass term $r > 0$ if the stationary solution satisfies the condition

$$0 < n_s < \min\left(\frac{T_0}{\varepsilon}, n_{s_+}\right).$$

If $T > T_c = T_0 + \varepsilon$ the stationary solution is stable in the whole accessible region $0 < n_s < T_0/\varepsilon$ whereas in the opposite case of low temperatures $T < T_c$ the stability against spatial fluctuations is only guaranteed in the restricted region $0 < n_s < n_{s_+}$. In case of a negative gap $r < 0$ spatial fluctuations leads to an instability against long wavelength fluctuations $\lambda > l^*$ where the critical wavelength l^* is defined by Eq. (33). From Eq. (35) one concludes that this instability occurs in the low-temperature regime $T < T_c$ where the accessible stationary density is restricted to the area $n_{s_+} < n_s < T_0/\varepsilon$. Summarizing both cases we get the final result: The stationary solution is achieved in the whole interval $0 < n_s < T_0/\varepsilon$ and it is stable against fluctuation if $T > T_c$. In the low-temperature case $T < T_c$ the stationary density is only stable at a low density $0 < n_s < n_{s_+}$ whereas the high density case $n_{s_+} < n_s < T_0/\varepsilon$ is unstable against fluctuations. The characteristic temperature

$$\frac{T_c}{\varepsilon} = 1 + \frac{\varepsilon}{2|J|z}$$

is originated by the mutual interaction. While a repulsive interaction between the bosons gives rise to an unstable behavior for wavelength $\lambda < l^*$ at each temperature, an attractive interaction leads to an instability against long wavelength fluctuations, however, only in limited temperature-density region. Above a characteristic temperature T_c the stationary solution is stable in the whole accessible density region $0 < n_s < T_0/\varepsilon$ whereas below T_c the system remains stable only in the low density regime $0 < n_s < n_{s_+} < T_0/\varepsilon$. The results are understandingly from a more heuristic point of view. If the particles are subject to a repulsive interaction they tend to maintain a fixed mutual distance. Therefore the effective energy to overcome the barrier between the particles is greater than the activation energy ε . The short range repulsive interaction leads to an instability on small length scale. The value l^* received by our analysis should be taken as a rough estimation for the size of the domains in which the system decays under the influence of the repulsive interaction. In case of an attractive interaction the effective energy to overcome a barrier is smaller than the activation en-

ergy. Due to the attraction the particles merge together and it is favorable that more particles surmount the barrier simultaneously. The system offers a tendency for a collapse even for finite temperatures. Thus, the stability is only guaranteed for a smaller density depending on temperature discussed in this section. Likewise the characteristic temperature should be considered as an estimation for the situation in mind. The behavior differs from the corresponding quantum case. In that case the Bose condensation temperature depends on the mass of the bosons $T_0 \propto m^{-1}$. Here, we have considered massless bosonic excitations and hence the ‘‘condensation’’ phenomenon is expected for high temperatures. A more detailed relation between quantum Bose particles and the stochastic dynamics, discussed here, remains an open problem.

VI. CONCLUSIONS

In the present paper we have discussed the counterpart of a spin-flip process in terms of Bose operators where a flip process means a creation or an annihilation of a particle at a certain lattice site. Different from the spins the occupation number per lattice site is not restricted. Accordingly the system must be coupled to an infinite particle reservoir. A particle has to overcome an energy barrier to leave the particle reservoir and to arrange it within the system. If the system is further coupled to a heat bath it tends to equilibrate. The stationary solution yields an averaged occupation number that is realized by the Bose distribution function. The model can be relevant for autocatalytic reactions combined with birth and death processes or for a deposition-desorption process. When a particle arrives at a random position on the system it forms a bond with the site, and sticks. Such a random local deposition process is realized with a certain rate γ whenever at least one particle is already present at that lattice site. The competing process is desorption occurring with the rate λ . The model can also be discussed in relation to the restricted growth of a bacterial colony that is influenced by external parameters as well as an internal interaction of the bacteria. The dynamical process is characterized by a relaxation time that shows a complete inverse behavior compared to the spin-flip case and to the conventional quantum case. If a short range density-density interaction is introduced the repulsive and the attractive interaction give rise to a complete different behavior. Whereas the repulsive interaction leads to an instability for short wave fluctuations, the attraction offers the tendency to a collapse at finite temperatures and for long wave fluctuations. The behavior of the system differs from the quantum case in a significant manner due to the fact that we have nonmassive Bose-like excitations. From here we conclude a similar phenomenon like Bose condensation in the high temperature limit. The analysis can be extended to hopping processes and to a random energy landscape in a manner which had been discussed elsewhere [25].

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